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Vectors fields, line integrals and surface integrals
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## Chapter 1

## Vectors field

Opening problem:
Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts. Shown in Figure 1.1


Figure 1.1:
is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station.

These applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without
an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

### 1.1 Vectors Fields in Space

Examples of vector fields. How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure 1.2 gives visual representations of such phenomena. 1.2 a shows a gravitational field exerted by two


Figure 1.2:
astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 1.2 b shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity
of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field.
Definition 1.1.1. A vector field in $\mathbb{R}^{3}$ is a function that assigns to each point $(x, y, z)$ a vector in space $\vec{F}=\vec{F}(x, y, x)$. The standard notation for the $\vec{F}$ function is:

$$
\vec{F}(x, y, z)=M(x, y, z) \vec{i}+N(x, y, z) \vec{j}+P(x, y, z) \vec{k},
$$

where $M, N$ and $P$ are differentiable scalar functions.

Example 1.1.1. Sketch each of the following vector field $\vec{F}(x, y, z)=2 x \vec{i}-$ $2 y \vec{j}-2 x \vec{k}$.

We made the sketching in Mathematica.


Figure 1.3:

Example 1.1.2. Vector field $\vec{v}=4|x| \vec{i}+\vec{j}$ models the velocity of water on the surface of a river. What is the speed of the water at point $A(2,3)$ ? Use meters per second as the units.

The vector field $\vec{v}$ at $A(2,3)$ will be $\vec{v}_{A}=8 \vec{i}+\vec{j}$. The speed of the water at this point is the magnitude of this vector, i.e., speed $=\left|\vec{v}_{A}\right|=\sqrt{8^{2}+1^{2}}=$ $\sqrt{65}=8.06 \mathrm{~m} / \mathrm{s}$.

## Describing a Gravitational Vector Field.

Newton's law of gravitation states that $\vec{F}=-G \frac{m_{1} m_{2}}{r^{2}} \vec{r}$ where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1 ) of mass $m_{1}$ located at the origin on another object of mass $m_{2}$ located at point $(x, y, z)$. Field $\vec{F}$ denotes the gravitational force that object 1 exerts on object $2, r$ is the distance between two objects, and $\vec{r}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. We will sketch the vector field associated with this equation.

We locate the object 1 in the origin. Now the distance between object 1 and object 2 is $r=\sqrt{x^{2}+y^{2}+z^{2}}$, and the unit vector is $\vec{r}=$ $\frac{1}{\sqrt{x^{2}+y^{2}+z^{2}}}(x \vec{i}+y \vec{j}+z \vec{k})$. Therefor the gravitational vector field is:
$\vec{F}=-G \frac{m_{1} m_{2}}{r^{2}} \vec{r}=-G \frac{m_{1} m_{2}}{r^{2}}\left(\frac{x}{r} \vec{i}+\frac{y}{r} \vec{j}+\frac{z}{r} \vec{k}\right)=-G \frac{m_{1} m_{2}}{r^{3}}(x \vec{i}+y \vec{j}+z \vec{k})$.
On the Figure 1.4 the gravitational vector field is given, sketched in Mathematica. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.
Example 1.1.3. The mass of asteroid 1 is $750,000 \mathrm{~kg}$ and the mass of asteroid 2 is $130,000 \mathrm{~kg}$. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15,-5,10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G=$ $6.67384 \times 10^{-11} \mathrm{~m}^{3} \mathrm{~kg}^{-1} \mathrm{~s}^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Gradient Vector Field. On of the "famous" vector field for a scalar function $f$ is its gradient vector field,

$$
\begin{equation*}
\nabla f=\frac{\partial f}{\partial x} \vec{i}+\frac{\partial f}{\partial y} \vec{j}+\frac{\partial f}{\partial z} \vec{k} \tag{1.2}
\end{equation*}
$$

These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational


Figure 1.4:
fields and electric fields associated with a static charge are examples of gradient fields.

Example 1.1.4. Find the gradient vector field for the function $f(x, y, z)=$ $z e^{-x y}$.

Using (1.2) we have:

$$
\nabla f=\frac{\partial f}{\partial x} \vec{i}+\frac{\partial f}{\partial y} \vec{j}+\frac{\partial f}{\partial z} \vec{k}=-y z e^{-x y} \vec{i}+-x z e^{-x y} \vec{j}+e^{-x y} \vec{k} .
$$

## Divergence and Curl

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higherdimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. For example, If $\vec{F}$ represents the velocity of a fluid, then the divergence of $\vec{F}$ at point $P$ measures the net rate
of change with respect to time of the amount of fluid flowing away from $P$ (the tendency of the fluid to flow "out of" $P$ ). In particular, if the amount of fluid flowing into $P$ is the same as the amount flowing out, then the divergence at $P$ is zero.

One application for divergence occurs in physics, when working with magnetic fields. Physicists use divergence in Gauss's law for magnetism which states that if $\vec{B}$ is a magnetic field, then its divergence is zero.

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. In other words, the curl at a point is a measure of the vector field's "spin" at a point.
Definition 1.1.2. Let $\vec{F}(x, y, z)=M(x, y, z) \vec{i}+N(x, y, z) \vec{j}+P(x, y, z) \vec{k}$ be a vector field, where $M, N$ and $P$ are differential functions. The divergence of $\vec{F}$ is the scalar function

$$
\operatorname{div} \vec{F}=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z},
$$

and curl of $\vec{F}$ is the vector field

$$
\operatorname{curl} \vec{F}=\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right) \vec{i}+\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right) \vec{j}+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \vec{k} .
$$

We define an operator $\nabla$ with

$$
\nabla=\frac{\partial}{\partial x} \vec{i}+\frac{\partial}{\partial y} \vec{j}+\frac{\partial}{\partial z} \vec{k} .
$$

Note that the divergence of $\vec{F}$ is the dot product of $\nabla$ and $\vec{F}$ :
$\operatorname{div} \vec{F}=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}=\left(\frac{\partial}{\partial x} \vec{i}+\frac{\partial}{\partial y} \vec{j}+\frac{\partial}{\partial z} \vec{k}\right) \cdot(M \vec{i}+N \vec{j}+P \vec{k})=\nabla \cdot \vec{F}$.
The curl of $\vec{F}$ is vector product of $\nabla$ and $\vec{F}$ : ,

$$
\begin{aligned}
\operatorname{curl} \vec{F} & =\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right) \vec{i}+\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right) \vec{j}+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \vec{k} \\
& =\left(\frac{\partial}{\partial x} \vec{i}+\frac{\partial}{\partial y} \vec{j}+\frac{\partial}{\partial z} \vec{k}\right) \times(M \vec{i}+N \vec{j}+P \vec{k})=\nabla \times \vec{F}
\end{aligned}
$$

Example 1.1.5. Find the Divergence and Curl of the vector field $\vec{F}(x, y, z)=$ $x^{2} y \vec{i}+x y z \vec{j}+\left(y^{2}+z^{2}\right) \vec{k}$.

We have:

$$
\begin{aligned}
\operatorname{div} \vec{F} & =\nabla \cdot \vec{F} \\
& =\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) \cdot\left(x^{2} y \vec{i}+x y z \vec{j}+\left(y^{2}+z^{2}\right) \vec{k}\right) \\
& =\frac{\partial}{\partial x}\left(x^{2} y\right)+\frac{\partial}{\partial y}(x y z)+\frac{\partial}{\partial z}\left(y^{2}+z^{2}\right) \\
& =2 x y+x z+2 z .
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{curl} \vec{F}= & \nabla \times \vec{F} \\
= & \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) \times\left(x^{2} y \vec{i}+x y z \vec{j}+\left(y^{2}+z^{2}\right) \vec{k}\right) \\
= & {\left[\frac{\partial}{\partial y}\left(y^{2}+z^{2}\right)-\frac{\partial}{\partial z}(x y z)\right] \vec{i}-\left[\frac{\partial}{\partial x}\left(y^{2}+z^{2}\right)-\frac{\partial}{\partial z}\left(x^{2} y\right)\right] \vec{j}+} \\
& +\left[\frac{\partial}{\partial x}(x y z)-\frac{\partial}{\partial y}\left(x^{2} y\right)\right] \vec{k} \\
= & (2 y-x y) \vec{i}+\left(y z-x^{2}\right) \vec{k}
\end{aligned}
$$

### 1.2 Line Integrals

In this section we are now going to introduce a new kind of integral, the line integrals. A line integral (sometimes called a path integral) of a scalar-valued function can be thought of as a generalization of the one-variable integral of a function over an interval, where the interval can be shaped into a curve. A simple analogy that captures the essence of a scalar line integral is that of calculating the mass of a wire from its density. A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space.

Since line integrals are integrals of a function over a curve, in the following we will assume that the curve is smooth, and is given by the parametric
equations, $x=x(t), y=y(t), z=z(t), \quad a \leq t \leq b$. We will often want to write the parameterizations of the curve as a vector function. In this case the curve is given by,

$$
\vec{r}(t)=x(t) \vec{i}+y(t) \vec{j}+z(t) \vec{k}, \quad a \leq t \leq b
$$

Scalar line integrals. This type of integral we also call it line integral of $f$ with respect to arc length.

For a formal description of a scalar line integral, let $C$ be a smooth curve in space given by the parameterization $r(\vec{t})=x(t) \vec{i}+y(t) \vec{j}+z(t) \vec{k}, \quad a \leq$ $t \leq b$. Let $f=f(x, y, z)$ be a function with a domain that includes curve $C$. To define the integral we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a, b]$ into $n$ subintervals $\left[t_{i}, t_{i+1}\right.$ ] where $t_{0}=a$ and $t_{n}=b$. Let $t_{i}^{*}$ be a value in the subinterval $\left[t_{i}, t_{i+1}\right]$. We denote the endpoints $\vec{r}\left(t_{0}\right), \vec{r}\left(t_{1}\right), \ldots, \vec{r}\left(t_{n}\right)$ with $P_{0}, P_{1}, \ldots, P_{n}$, see figure 1.5. Points $P_{i}$ divide the curve $C$ into pieces $C_{0}, C_{1}, \ldots, C_{n}$, with length $\triangle s_{0}, \triangle s_{1}, \ldots, \triangle s_{n}$. At the end, we evaluate the function for the point $P_{i}^{*}$, multiply with $\triangle s_{i}$, and sum for $1 \leq i \leq n$.


Figure 1.5:
Definition 1.2.1. Let $f$ be a function with a domain that includes the smooth curve $C$ that is parameterized by $\vec{r}(t)=x(t) \vec{i}+y(t) \vec{j}+z(t) \vec{k}, \quad a \leq t \leq b$. The scalar line integral of $f$ along $C$ is:

$$
\int_{C} f(x, y, z) d s=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} f\left(P_{i}^{*}\right) \triangle s_{i}
$$

if this limit exists and $\triangle s_{i}$ and $P_{i}^{*}$ are defined as before.

If $f$ is a continuous function on a smooth curve $C$ then the integral exists. Next, we will show how to evaluate the scalar line integrals.
Theorem 1.2.1. Let $f$ be a continuous function with a domain that includes the smooth curve $C$ with parameterizations $\vec{r}(t)=x(t) \vec{i}+y(t) \vec{j}+z(t) \vec{k}, \quad a \leq$ $t \leq b$. Then

$$
\int_{C} f(x, y, z) d s=\int_{a}^{b} f(x(t), y(t), z(t)) \sqrt{\left[x^{\prime}(t)\right]^{2}+\left[y^{\prime}(t)\right]^{2}+\left[z^{\prime}(t)\right]^{2}} d t
$$

Example 1.2.1. Find $\int_{C}\left(x^{2}+y^{2}+z^{2}\right) d s$ where $C$ is given by $x=\cos t, y=$ $\sin t, z=t$ for $0 \leq t \leq 2 \pi$, from $(1,0,0)$ to $(1,0,2 \pi)$.

The curve $C$ is given of figure 1.6. We have:


Figure 1.6:

$$
\begin{aligned}
\int_{C}\left(x^{2}+y^{2}+z^{2}\right) d s & =\int_{0}^{2 \pi}\left(\cos ^{2} t+\sin ^{2} t+t^{2}\right) \sqrt{(-\sin t)^{2}+(\cos t)^{2}+1^{2}} d t \\
& =\int_{0}^{2 \pi}\left(1+t^{2}\right) \sqrt{2} d t=2 \sqrt{2} \pi\left(1+4 \pi^{2} / 3\right)
\end{aligned}
$$

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x, y, z)=1$ then the arc length of $C$ is given by $\int_{C} d s$.
Example 1.2.2. Find the length of a wire with parameterization $\vec{r}(t)=$ $(3 t+1) \vec{i}+(4-2 t) \vec{j}+(5+2 t) \vec{k}, \quad 0 \leq t \leq 4$.

We have:

$$
\begin{aligned}
\int_{C} d s & =\int_{0}^{4} \sqrt{(3)^{2}+(-2)^{2}+2^{2}} d t \\
& =\int_{0}^{4} \sqrt{17} d t=4 \sqrt{17}
\end{aligned}
$$

Since $\triangle s_{i}>0$, when we switch the direction of the curve the line integral (with respect to arc length) will not change. The following property hold: $\int_{C} f(x, y, z) d s=\int_{-C} f(x, y, z) d s$. The line integral has the following properties:

$$
\begin{aligned}
\int_{C}(f \pm g)(x, y, z) d s & =\int_{C} f(x, y, z) d s \pm \int_{C} g(x, y, z) d s \\
\int_{C} k f(x, y, z) d s & =k \int_{C} f(x, y, z) d s, \quad \text { for a constant } k .
\end{aligned}
$$

Evaluation of line integrals over piecewise smooth curves is a relatively simple thing to do. All we do is evaluate the line integral over each of the pieces and then add them up. The line integral for some function over a piecewise curve would be

$$
\int_{C} f(x, y, z) d s=\int_{C_{1}} f(x, y, z) d s+\int_{C_{2}} f(x, y, z) d s
$$

A mass of a wire. Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Here, we calculate the mass of a wire using a scalar line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x, y, z)$. We can calculate the total mass of the wire using the scalar line integral $\int_{C} \rho(x, y, z) d s$.
Example 1.2.3. Calculate the mass of a spring in the shape of a curve parameterized by $\vec{r}(t)=t \vec{i}+2 \cos t \vec{j}+2 \sin t \vec{k}, \quad 0 \leq t \leq \frac{\pi}{2}$, with a density function given by $\rho(x, y, z)=e^{x}+y z$.

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_{C} \rho(x, y, z) d s$. We have

$$
\begin{aligned}
\int_{C} \rho(x, y, z) d s & =\int_{0}^{\pi / 2}\left(e^{x}+y z\right) \sqrt{x^{\prime 2}(t)+y^{\prime 2}(t)+z^{\prime 2}(t)} d t= \\
& =\int_{0}^{\pi / 2}\left(e^{t}+4 \sin t \cos t\right) \sqrt{1^{2}+(-2 \sin t)^{2}+(2 \cos t)^{2}} d t= \\
& =\sqrt{5} \int_{0}^{\pi / 2}\left(e^{t}+4 \sin t \cos t\right) d t=\sqrt{5}\left(e^{\pi / 2}+1\right)
\end{aligned}
$$

Vector line integrals. The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. Let $\vec{F}(x, y, z)=M(x, y, z) \vec{i}+N(x, y, z) \vec{j}+P(x, y, z) \vec{k}$, be a continued vector field in $\mathbb{R}^{3}$ that represents a force on a particle, and let $C$ be a smooth curve in $\mathbb{R}^{3}$ contained in the domain of $\vec{F}$. How would we compute the work done by $\vec{F}$ in moving a particle along $C$ ? We need a vector line integral.
Definition 1.2.2. Let $f$ be a function with a domain that includes the smooth curve $C$ that is parameterized by $\vec{r}(t)=x(t) \vec{i}+y(t) \vec{j}+z(t) \vec{k}, \quad a \leq t \leq b$. The vector line integral of $\vec{F}$ along $C$ is:

$$
\begin{equation*}
\int_{C} \vec{F} \cdot d \vec{r}(t)=\int_{a}^{b} \vec{F}(\vec{r}(t)) \vec{r}^{\prime}(t) d t \tag{1.3}
\end{equation*}
$$

In this notation we have $\vec{r}^{\prime}(t)=x^{\prime}(t) \vec{i}+y^{\prime}(t) \vec{j}+z^{\prime}(t) \vec{k}$.
Therefore, the work done by $\vec{F}$ in moving the particle in the positive direction along $C$ is defined as:

$$
\begin{equation*}
W=\int_{C} \vec{F} \cdot d \vec{r}(t) \tag{1.4}
\end{equation*}
$$

The vector line integral has the following properties:

$$
\begin{aligned}
\int_{-C} \vec{F} \cdot d \vec{r}(t) & =-\int_{C} \vec{F} \cdot d \vec{r}(t) \\
\int_{C}(\vec{F}+\vec{G}) \cdot d \vec{r}(t) & =\int_{C} \vec{F} \cdot d \vec{r}(t)+\int_{C} \vec{G} \cdot d \vec{r}(t) \\
\int_{C} k \vec{F} \cdot d \vec{r}(t) & =k \int_{C} \vec{F} \cdot d \vec{r}(t), \quad \text { for a constant } k
\end{aligned}
$$

Similar as in the scalar line integrals, the evaluation of vector line integrals over piecewise smooth curves would be

$$
\int_{C} \vec{F} \cdot d \vec{r}(t)=\int_{C_{1}} \vec{F} \cdot d \vec{r}(t)+\int_{C_{2}} \vec{F} \cdot d \vec{r}(t)
$$

Example 1.2.4. Find the value of the vector line integral $\int_{C} \vec{F} \cdot d \vec{r}(t)$ where $C$ is the circle parameterized with $\vec{r}(t)=\cos t \vec{i}+\sin t \vec{j}, \quad 0 \leq t \leq \pi$, and $\vec{F}=-y \vec{i}+x \vec{j}$.

Note that in this example the vector field id two dimensional. Using (1.3) we have:

$$
\begin{aligned}
\int_{C} \vec{F} \cdot d \vec{r}(t) & =\int_{a}^{b} \vec{F}(\vec{r}(t)) \vec{r}^{\prime}(t) d t \\
& =\int_{0}^{\pi}(-\sin t \vec{i}+\cos t \vec{j}) \cdot(-\sin t \vec{i}+\cos t \vec{j}) d t= \\
& =\int_{0}^{\pi}\left(\sin ^{2} t+\cos ^{2} t\right) d t=\int_{0}^{\pi} d t=\pi
\end{aligned}
$$

We can rewrite the left hand side of equation (1.3) so that it becomes:

$$
\int_{C} \vec{F} \cdot d \vec{r}(t)=\int_{C} M d x+N d y+P d z
$$

This is another standard notation for the vector line integral.
Example 1.2.5. Find $\int_{C} x^{2} y d x+(y-z) d y+x z d z$ where:
(a) $C$ is the curve given with the equations $\vec{r}(t)=\vec{i}+t^{2} \vec{j}+t \vec{k}, \quad 0 \leq t \leq 2$;
(b) $C$ is the line from $(1,0,0)$ to $(1,4,2)$.
(a) For $\vec{r}(t)=\vec{i}+t^{2} \vec{j}+t \vec{k}$ we have $x(t)=1, y(t)=t^{2} \mathrm{i} z(t)=t$, and $d x=0 \cdot d t$, $d y=2 t d t, d z=d t$. So

$$
x^{2} y d x+(y-z) d y+x z d z=1^{2} t^{2} \cdot 0 \cdot d t+\left(t^{2}-t\right) 2 t d t+1 \cdot t d t
$$

For the line integral we have:

$$
\int_{C} x^{2} y d x+(y-z) d y+x z d z=\int_{0}^{2}\left(2 t^{3}-2 t^{2}+t\right) d t=\frac{14}{3} .
$$

(b) The line passing trough $(1,0,0)$ and $(1,4,2)$ has the parametric equations $x(t)=1, y(t)=4 t$ and $z(t)=2 t$ where $0 \leq t \leq 1$, and $d x=0 \cdot d t, d y=4 d t$, $d z=2 d t$. We have

$$
x^{2} y d x+(y-z) d y+x z d z=1^{2} 4 t \cdot 0 d t+(4 t-2 t) 4 d t+2 t \cdot 2 d t
$$

For the line integral in this case we have:

$$
\int_{C} x^{2} y d x+(y-z) d y+x z d z=\int_{0}^{1} 12 t d t=6
$$

Both curves are given of figure 1.7. They start at the same point $(1,0,0)$, and end at the same point $(1,4,2)$. But the value of the integral is not the same.


Figure 1.7:

A work done by a force. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here we calculate the work done by a force using a vector line integral.

Example 1.2.6. How much work is required to move an object in vector force field $\vec{F}=y z \vec{i}+x y \vec{j}+x z \vec{k}$ along path $\vec{r}(t)=t^{2} \vec{i}+t \vec{j}+t^{4} \vec{k}, 0 \leq t \leq 1$.

From the equation (1.4) we have:

$$
\begin{aligned}
W=\int_{C} \vec{F} \cdot d \vec{r}(t) & =\int_{0}^{1}(y z \vec{i}+x y \vec{j}+x z \vec{k}) \cdot\left(2 t \vec{i}+\vec{j}+4 t^{3} \vec{k}\right) d t= \\
& =\int_{0}^{1}\left(t^{5} \cdot 2 t+t^{3}+t^{6} \cdot 4 t^{3}\right) d t=\frac{131}{140}
\end{aligned}
$$

Line integrals as circulation. The vector line integral introduction explains how the line integral $\int_{C} \vec{F} \cdot d \vec{r}(t)$ of a vector field $\vec{F}$ over an oriented curve $C$ "adds up" the component of the vector field that is tangent to the curve. In this sense, the line integral measures how much the vector field is aligned with the curve. If the curve $C$ is a closed curve, then the line integral indicates how much the vector field tends to circulate around the curve $C$. In fact, for an oriented closed curve $C$, we call the line integral the "circulation" of $\vec{F}$ around $C$, see figure 1.8. To emphasize that the integral is around a closed curve we use the $\oint$ notation.

$$
\oint_{C} \vec{F} \cdot d \vec{r}(t)=\text { circulation of } \vec{F} \text { over } C
$$



Figure 1.8:

Example 1.2.7. Find the circulation of the vector field $\vec{F}=y \vec{i}-x \vec{j}$ over the ellipse $\frac{x^{2}}{4}+\frac{y^{2}}{9}=1$.

First we write the ellipse with its vector form (using parametric equations): $\vec{r}(t)=2 \cos t \vec{i}+3 \sin t \vec{j} 0 \leq t \leq 2 \pi$. To find the circulation we just
evaluate the vector line integral. We got:

$$
\begin{aligned}
\oint_{C} \vec{F} \cdot d \vec{r}(t) & =\int_{0}^{2 \pi}(y \vec{i}-x \vec{j}) \cdot(-2 \sin t \vec{i}+3 \cos t \vec{j}) d t= \\
& =\int_{0}^{2 \pi}(3 \sin t \vec{i}-2 \cos t \vec{j}) \cdot(-2 \sin t \vec{i}+3 \cos t \vec{j}) d t \\
& =\int_{0}^{2 \pi}\left(-6 \sin ^{2} t-6 \cos ^{t}\right)=-12 \pi
\end{aligned}
$$

As shown in figure 1.8, the vector field appears to circulate in the clockwise direction, tending to point in the opposite direction of the orientation of the curve, and we got negative circulation.

### 1.3 Conservative Vectors Fields

A vector field $\vec{F}$ is called a conservative vector field if there exists a scalar function $f$ such that $\vec{F}=\nabla f$. If $\vec{F}$ is a conservative vector field then the function, $f$, is called a potential function for $\vec{F}$. All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

For instance the vector field $\vec{F}=y \vec{i}+x \vec{j}$ is a conservative vector field with a potential function of $f(x, y)=x y$ because $\nabla f=y \vec{i}+x \vec{j}=\vec{F}$.
Example 1.3.1. Is $f(x, y, z)=x^{2} y z-\sin (x y)$ a potential function for vector field $\vec{F}(x, y, z)=\left(\begin{array}{c}2 x y z-y \cos (x y) \\ x^{2} z-x \cos (x y) \\ x^{2} y\end{array}\right)$ ?

We need to confirm whether $\vec{F}=\nabla f$. We have

$$
\begin{aligned}
f_{x}^{\prime} & =2 x y z-y \cos (x y) \\
f_{y}^{\prime} & =x^{2} z-x \cos (x y) \\
f_{z}^{\prime} & =x^{2} y .
\end{aligned}
$$

Therefore, $\nabla f=\vec{F}$, and $f$ is a potential function for $\vec{F}$.
If $\vec{F}$ is a conservative vector field, then there is at least one potential function $f$ such that $\nabla f=\vec{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for
the same vector field? It is proved that if $\vec{F}$ is a conservative vector field on an open and connected domain and let $f$ and $g$ are functions such that $\nabla f=\vec{F}$, and $\nabla g=\vec{F}$, then there is a constant $C$ such that $f=g+C$. So, for a conservative vector field, the potential functions differ for a constant. In a way there is an analogy between the potential functions and the antiderivative functions.

Many physical force fields (vector fields) that you are familiar with are conservative vector fields. The term comes from the fact that some kind of energy is conserved by these force fields. The important consequence for us, though, is that as you move an object from point $A$ to point $B$, the work performed by a conservative force field does not depend on the path taken from point $A$ to point $B$. For this reason, we often refer to such vector fields as path-independent vector fields. Path-independent and conservative are just two terms that mean the same thing. Not all vector fields are conservative. If a vector field is not path-independent, we call it path-dependent (or nonconservative). For example, the vector field $\vec{F}=x^{2} y \vec{i}+(y-z) \vec{j}+x z \vec{k}$ considered in the example 1.2.5 is path depended.

Next, we give the fundamental theorem for line integrals.
Theorem 1.3.1. Suppose that $C$ is a smooth curve given by $\vec{r}(t), a \leq t \leq b$. Also suppose that $f$ is a function whose gradient vector $\nabla f$ is continuous on C. Then

$$
\int_{C} \vec{\nabla} f \cdot d \vec{r}(t)=f(\vec{r}(b))-f(\vec{r}(a)) .
$$

We use this theorem to make a connection with the conservative vector fields. We know that if $\vec{F}$ is a conservative vector field, there is a potential function $f$ such that $\nabla f=\vec{F}$. Then:

$$
\int_{C} \vec{F} \cdot d \vec{r}(t)=\int_{C} \vec{\nabla} f \cdot d \vec{r}(t)=f(\vec{r}(b))-f(\vec{r}(a))
$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_{C} \vec{F} \cdot d \vec{r}(t)$ where $\vec{F}$ is a conservative vector field is a two steps process:

1. Find a potential function for $\vec{F}$ (the "antiderivative"; )
2. Compute the value of $f$ at the endpoints of $C$ and calculate their difference.

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vec{F}$ is conservative and $C$ is a closed curve, then the circulation of $\vec{F}$ along $C$ is zero-that is $\int_{C} \vec{F} \cdot d \vec{r}(t)=0$. The second important consequence of the Fundamental Theorem for Line Integrals is that line integrals of conservative vector fields are independent of path-meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

Finally, in this section we want to look at two questions. First, given a vector field $\vec{F}$ is there any way of determining if it is a conservative vector field? Secondly, if we know that $\vec{F}$ is a conservative vector field how do we go about finding a potential function for the vector field? To answer the first quaestion we have the following theorem.
Theorem 1.3.2. If $\vec{F}$ is defined on $\mathbb{R}^{3}$ whose components have continuous first order partial derivative and curl $\vec{F}=\overrightarrow{0}$, then $\vec{F}$ is a conservative vector field.

In the following example using Theorem 1.3 .2 we will prove that $\vec{F}$ is conservative vector field, and we will illustrate the procedure for finding its potential $f$.
Example 1.3.2. Show that

$$
\vec{F}(x, y, z)=2 x y \vec{i}+\left(x^{2}+\sin z\right) \vec{j}+(y \cos z+2) \vec{k}
$$

is conservative vector field, and find its potential.
To show that $\vec{F}$ is conservative vector field first we find $\operatorname{curl} \vec{F}$.

$$
\begin{aligned}
\operatorname{curl} \vec{F}= & \nabla \times \vec{F} \\
= & \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) \times\left(=2 x y \vec{i}+\left(x^{2}+\sin z\right) \vec{j}+(y \cos z+2) \vec{k}\right) \\
= & {\left[\frac{\partial}{\partial y}(y \cos z+2)-\frac{\partial}{\partial z}\left(x^{2}+\sin z\right)\right] \vec{i}-\left[\frac{\partial}{\partial x}(y \cos z+2)-\frac{\partial}{\partial z}(2 x y)\right] \vec{j}+} \\
& +\left[\frac{\partial}{\partial x}\left(x^{2}+\sin z\right)-\frac{\partial}{\partial y}(2 x y)\right] \vec{k} \\
= & (\cos z-\cos z) \vec{i}+(0-0) \vec{j}+(2 x-2 x) \vec{k}=\overrightarrow{0} .
\end{aligned}
$$

Since $\operatorname{curl} \vec{F}=0$, from Theorem 1.3.2 we have that $\vec{F}$ is conservative vector field. Let us find a potential $f$ such that

$$
\vec{F}=\nabla f=\frac{\partial f}{\partial x} \vec{i}+\frac{\partial f}{\partial y} \vec{j}+\frac{\partial f}{\partial z} \vec{k}
$$

From these equations we have

$$
\frac{\partial f}{\partial x}=2 x y, \quad \frac{\partial f}{\partial y}=x^{2}+\sin z, \quad \frac{\partial f}{\partial z}=y \cos z+2
$$

Integrating in respect to $x$ the first equation we have

$$
f(x, y, z)=\int 2 x y d x=x^{2} y+g(y, z)
$$

Next, we take a partial derivative with respect to $y$ and we have

$$
\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^{2} y+g(y, z)\right)=x^{2}+\frac{\partial}{\partial y}(g(y, z))=x^{2}+\sin z
$$

For $g(y, z)$ we got $g(y, z)=\int \sin z d y=y \sin z+k(z)$. So, $f$ will be $f(x, y, z)=$ $x^{2} y+g(y, z)=x^{2}+y \sin z+k(z)$, and now we take a partial derivative of $f$ with respect to $z$. We have

$$
\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\left(x^{2}+y \sin z+k(z)\right)=y \cos z+k^{\prime}(z)=y \cos z+2 .
$$

From $k^{\prime}(z)=2$ we got $k(z)=2 z+C$ and $f$ is found:

$$
f(x, y, z)=x^{2} y+y \sin z+k(z)=x^{2}+y \sin z+2 z+C .
$$

Example 1.3.3. Show that the gravitational force vector field

$$
\vec{F}=\frac{-G x}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}} \vec{i}+\frac{-G y}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}} \vec{j}+\frac{-G z}{\left(x^{2}+y^{2}+z^{2}\right)^{3 / 2}} \vec{k}
$$

is conservative and find its potential.

### 1.4 Surface Integrals

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration. Let's start off with a sketch of the surface $S$ since the notation can get a little confusing once we get into it. Here, on figure 1.9 is a sketch of some surface $S$. In this case the surface $S$


Figure 1.9:
lies above some region $D$ that lies in the $x 0 y$ plane. Also note that we could just as easily looked at a surface $S$ that was in front of some region $D$ in the $y 0 z$ plane or the $x 0 z$ plane.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. Surface integrals are similar to line integrals. Just as there are two types of integrals over curves (line integrals of scalar functions and of vector fields) there are two types of surface integrals: surface integrals of scalar functions, and surface integrals of vector fields. The surface integral of a scalar function is a simple generalization of a double integral. Like the
line integral of vector fields, the surface integrals of vector fields will play a big role in the fundamental theorems of vector calculus.

Surface integral of a scalar-valued function. The definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain region (no longer a curve) into tiny pieces and proceed in the same fashion. Let $S$ be a piecewise smooth surface with parameterizations $r(\vec{u}, v)=x(u, v) \vec{i}+y(u, v) \vec{j}+z(u, v) \vec{k}$, with parameter domain $D$ and let $f=f(x, y, z)$ be a function with a domain that contains $S$. We can assume that $D$ is a rectangle, and divide it into subrectangles $D_{i j}$, with horizontal width $\Delta u$ and vertical lenght $\Delta v$. This division of $D$ into subrectangles gives a corresponding division of $S$ into pieces $S_{i j}$. Choose point $P_{i j}$ in each piece $S_{i j}$, evaluate $f$ at $P_{i j}$, multiply the area of $S_{i j}$ and form the Rieman sum:

$$
\sum_{i=1}^{n} \sum_{i=1}^{n} f\left(P_{i j}\right) \triangle S_{i j} .
$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of $S$ shrink to zero by taking a limit.
Definition 1.4.1. The surface integral of a scalar-valued function of $f$ over a piecewise smooth surface $S$ is:

$$
\iint_{S} f(x, y, z) d \sigma=\lim _{n \rightarrow \infty} \sum_{i=1}^{n} \sum_{i=1}^{n} f\left(P_{i j}\right) \triangle S_{i j}
$$

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas $\triangle S_{i j}$ with small pieces of a tangent plane. We have:

$$
\triangle S_{i j} \approx\left|\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}\right| \triangle u \Delta v
$$

The surface integral is

$$
\iint_{S} f(x, y, z) d \sigma=\iint_{D} f(r(\vec{u}, v))\left|\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}\right| d P
$$

Let consider the case when the surface $S$ is given with the equation $z=$ $g(x, y)$. Then we can parameterize it with the equations $x=u$ and $y=v$, then $r(\vec{u}, v)=u \vec{i}+v \vec{j}+g(u, v) \vec{k}$. We have

$$
\left|\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}\right|=\left\|\begin{array}{ccc}
\vec{j} & \vec{j} & \vec{k} \\
1 & 0 & g_{u}^{\prime} \\
0 & 1 & g_{v}^{\prime}
\end{array}\right\|=\left|-g_{u}^{\prime} \vec{i}-g_{v}^{\prime} \vec{j}+\vec{k}\right|=\sqrt{\left(g_{u}^{\prime}\right)^{2}+\left(g_{v}^{\prime}\right)^{2}+1}
$$

and in this case we can use the following equation to find a surface integral of a scalar-valued function:

$$
\iint_{S} f(x, y, z) d \sigma=\iint_{D} f(x, y, g(x, y)) \sqrt{\left(g_{x}^{\prime}\right)^{2}+\left(g_{y}^{\prime}\right)^{2}+1} d P
$$

Example 1.4.1. Find $\iint_{S} z d \sigma$, where $S$ is the surface $z=\sqrt{4-x^{2}-y^{2}}$.
The surface $S$ is half of a sphere with $r=2$, see figure 1.10. The region


Figure 1.10:
$D$ in the $x 0 y$ plane $x^{2}+y^{2} \leq 4$, and we have:

$$
\begin{aligned}
\iint_{S} z d \sigma & =\iint_{D} \sqrt{4-x^{2}-y^{2}}\left\{\left(\frac{\partial \sqrt{4-x^{2}-y^{2}}}{\partial x}\right)^{2}+\left(\frac{\partial \sqrt{4-x^{2}-y^{2}}}{\partial y}\right)^{2}+1\right\}^{1 / 2} d P \\
& =\int_{-2}^{2} \int_{-\sqrt{4-x^{2}}}^{\sqrt{4-x^{2}}} \sqrt{4-x^{2}-y^{2}}\left[\frac{x^{2}}{4-x^{2}-y^{2}}+\frac{y^{2}}{4-x^{2}-y^{2}}+1\right]^{1 / 2} d y d x \\
& =\int_{-2}^{2} \int_{-\sqrt{4-x^{2}}}^{\sqrt{4-x^{2}}} 2 d y d x .
\end{aligned}
$$

Using polar coordinates we find:

$$
\iint_{S} z d \sigma=2 \int_{0}^{2 \pi} \int_{0}^{2} r d r d \theta=8 \pi
$$

A mass of a sheet. Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point $(x, y, z)$ is $\rho(x, y, z)$ then mass $m$ of the sheet is:

$$
m=\iint_{S} \rho(x, y, z) d \sigma
$$

Example 1.4.2. A piece of metal has a shape that is modeled by paraboloid $z=x^{2}+y^{2}, 0 \leq z \leq 4$, and the density of the metal is given by $\rho(x, y, z)=$ $z+1$. Find the mass of the piece of metal.

To find the mass of the piece of metal, we will find the scalar surface integral: $m=\iint_{S} \rho(x, y, z) d \sigma=\iint_{S}(z+1) d \sigma$. We have:

$$
\begin{aligned}
\iint_{S}(z+1) d \sigma & =\iint_{D}\left(x^{2}+y^{2}+1\right)\left\{\left(\frac{\partial\left(x^{2}+y^{2}\right)}{\partial x}\right)^{2}+\left(\frac{\partial\left(x^{2}+y^{2}\right)}{\partial y}\right)^{2}+1\right\}^{1 / 2} d P \\
& =\int_{-2}^{2} \int_{-\sqrt{4-x^{2}}}^{\sqrt{4-x^{2}}}\left(x^{2}+y^{2}+1\right)\left[4\left(x^{2}+y^{2}\right)+1\right]^{1 / 2} d y d x= \\
& =\frac{1}{20}(-3+187 \sqrt{17}) \pi
\end{aligned}
$$

Surface integral of a vector field. The line integral of a vector field $\vec{F}$ could be interpreted as the work done by the force field $\vec{F}$ on a particle moving along the path. The surface integral of a vector field $\vec{F}$ actually has a simpler explanation. If the vector field $\vec{F}$ represents the flow of a fluid, then the surface integral of $\vec{F}$ will represent the amount of fluid flowing through the surface (per unit time).

The amount of the fluid flowing through the surface per unit time is also called the flux of fluid through the surface. For this reason, we often call the surface integral of a vector field a flux integral.

If water is flowing perpendicular to the surface, a lot of water will flow through the surface and the flux will be large. On the other hand, if water is flowing parallel to the surface, water will not flow through the surface, and the flux will be zero. To calculate the total amount of water flowing through the surface, we want to add up the component of the vector $\vec{F}$ that is perpendicular to the surface. The direction of this perpendicular vector is very important since it gives the orientation of the surface $S$. When we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation. Let us explain this more accurately.

Let $S$ be a smooth surface. For any point $(x, y, z)$ on $S$, we can identify two unit normal vectors $\vec{n}$ and $-\vec{n}$. If it is possible to choose a unit normal vector $\vec{n}$ at every point $(x, y, z)$ on $S$ so that $\vec{n}$ varies continuously over $S$, then $S$ is "orientable" surface. Such a choice of unit normal vector at each point gives the orientation of a surface $S$. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives $S$ an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

For example, closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the positive orientation of the closed surface, see figure 1.11. We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

Let $\vec{n}$ be a unit normal vector to the surface. The choice of normal vector orients the surface and determines the sign of the fluid flux. The flux of fluid through the surface is determined by the component of $\vec{F}$ that is in the direction of $\vec{n}$, i.e. by $\vec{F} \cdot \vec{n}$.

Now, we need to discuss how to find the unit normal vector if the surface is given perimetrically as, $r(\overrightarrow{u, v})=x(u, v) \vec{i}+y(u, v) \vec{j}+z(u, v) \vec{k}$. In this case the vector $\vec{r}_{u}{ }^{\prime} \times \vec{r}_{v}{ }^{\prime}$ will be normal to the tangent plane at a particular point. But if the vector is normal to the tangent plane at a point then it will also be normal to the surface at that point. So, this is a normal vector. In order to guarantee that it is a unit normal vector we will also need to divide it by


Figure 1.11:
its magnitude.
So, in the case of parametric surfaces one of the unit normal vectors will be,

$$
\vec{n}=\frac{\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}}{\left|\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}\right|} .
$$

We will need to look at this once it's computed and determine if it points in the correct direction or not. If it doesn't then we can always take the negative of this vector and that will point in the correct direction.

Let us now have a surface given with the equation $z=g(x, y)$. Then, as before expalined, we can parameterize it with the equations $x=u$ and $y=v$, then $r(\vec{u}, v)=u \vec{i}+v \vec{j}+g(u, v) \vec{k}$, and $\vec{r}_{u}{ }^{\prime} \times \vec{r}_{v}{ }^{\prime}=-g_{u}^{\prime} \vec{i}-g_{v}^{\prime} \vec{j}+\vec{k}$, so for the $\vec{n}$ we have:

$$
\vec{n}=\frac{\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}}{\left|\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}\right|}=\frac{-g_{x}^{\prime} \vec{i}-g_{y}^{\prime} \vec{j}+\vec{k}}{\sqrt{\left(g_{x}^{\prime}\right)^{2}+\left(g_{y}^{\prime}\right)^{2}+1}}
$$

We finally have the definition of the surface integral of a vector field.
Definition 1.4.2. Let $\vec{F}$ be a continuous vector field with a domain that contains oriented surface $S$ with unit normal vector $\vec{n}$. The surface integral of $\vec{F}$ over $S$ is

$$
\iint_{S} \vec{F} \cdot d \vec{S}=\iint_{S} \vec{F} \cdot \vec{n} d \sigma
$$

The second surface integral in the above definition 1.4.2 is a surface integral of a scalar valued function, and its connection with a double integral is given by the equation:

$$
\iint_{S} \vec{F} \cdot \vec{n} d \sigma=\iint_{D} \vec{F} \cdot\left(\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}\right) d P
$$

where surface is given perimetrically as, $r(\vec{u}, v)=x(u, v) \vec{i}+y(u, v) \vec{j}+z(u, v) \vec{k}$. So it is easier to find the surface integral when $S$ is given by its parametric equations.

In the other case, let the surface be given by $z=g(x, y)$ In this case let's also assume that the vector field is given by $\vec{F}=M \vec{i}+N \vec{j}+P \vec{k}$, and that the orientation that we are after is the "upwards" orientation. Under all of these assumptions the surface integral of $\vec{F}$ over $S$ is:

$$
\begin{aligned}
\iint_{S} \vec{F} \cdot d \vec{S} & =\iint_{S} \vec{F} \cdot \vec{n} d \sigma= \\
& =\iint_{D}(M \vec{i}+N \vec{j}+P \vec{k}) \cdot\left(\frac{-g_{x}^{\prime} \vec{i}-g_{y}^{\prime} \vec{j}+\vec{k}}{\sqrt{\left(g_{x}^{\prime}\right)^{2}+\left(g_{y}^{\prime}\right)^{2}+1}}\right) \\
& \sqrt{\left(g_{x}^{\prime}\right)^{2}+\left(g_{y}^{\prime}\right)^{2}+1} d P= \\
& =\iint_{D}\left(-M g_{x}^{\prime}-N g_{y}^{\prime}+P\right) d P .
\end{aligned}
$$

Now, remember that this assumed the "upward" orientation. If we would needed the "downward" orientation, then we would need to change the signs on the normal vector. This would in turn change the signs on the integrand as well. So, we really need to be careful here when using this formula. In general, it is best to rederive this formula as you need it.

Example 1.4.3. Let $S$ be the cylinder of radius 3 and height 5 given by $x^{2}+y^{2}=9,0 \leq z \leq 5$. Let $\vec{F}$ be a vector field, $\vec{F}=2 x \vec{i}+2 y \vec{j}+2 z \vec{k}$. Find the integral of $\vec{F}$ over $S$, where $S$ is the positive side outside of the cylinder, i.e., we will use the outward pointing normal vector, see figure 1.12.

To find the integral, first we make a parametrization of the cylinder. We have $r(\vec{u}, v)=3 \cos u \vec{i}+3 \sin u \vec{j}+v \vec{k}$, for $0 \leq u \leq 2 \pi, 0 \leq v \leq 5$.


Figure 1.12:

Next we find the normal vector to the surface $\vec{r}_{u}{ }^{\prime} \times \vec{r}_{v}{ }^{\prime}$ :

$$
\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}=\left|\begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
-3 \sin u & 3 \cos u & 0 \\
0 & 0 & 1
\end{array}\right|=3 \cos u \vec{i}+3 \sin u \vec{j},
$$

and as shown in the below figure 1.13 it is an outward pointing normal. We


Figure 1.13:
can now calculate total flux.

$$
\begin{aligned}
\iint_{S} \vec{F} \cdot d \vec{S} & =\iint_{S} \vec{F} \cdot \vec{n} d \sigma= \\
& =\iint_{S}(2 x \vec{i}+2 y \vec{j}+2 z \vec{k}) \cdot(3 \cos u \vec{i}+3 \sin u \vec{j}) \frac{1}{3} d \sigma= \\
& =\iint_{D}(6 \cos u \vec{i}+6 \sin u \vec{j}+2 v \vec{k}) \cdot(3 \cos u \vec{i}+3 \sin u \vec{j}) \frac{1}{3} \cdot 3 d P= \\
& =\iint_{D}\left(18 \cos ^{2} u+18 \sin ^{2} u\right) d P= \\
& =\int_{0}^{2 \pi} \int_{0}^{5} 18 d v d u=180 \pi
\end{aligned}
$$

Example 1.4.4. Let $S$ be a disk of radius 6 centered around the $z$ axis in plane $z=-4$, oriented with an upward pointing normal. Let a magnetic field be given by $\vec{F}=\left(x^{2}+y^{2}\right) \vec{k}$. What is total magnetic flux through disk?

Let us first parametrize the given surface $S$. We have $r(\vec{u}, v)=v \cos u \vec{i}+$ $v \sin u \vec{j}+(-4) \vec{k}$, for $0 \leq u \leq 2 \pi, 0 \leq v \leq 6$. Next we find the normal vector to the surface $\vec{r}_{u}{ }^{\prime} \times \vec{r}_{v}{ }^{\prime}$ :

$$
\vec{r}_{u}^{\prime} \times \vec{r}_{v}^{\prime}=\left|\begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
-v \sin u & v \cos u & 0 \\
\cos u & \sin u & 1
\end{array}\right|=-v \vec{k} .
$$

The normal vector is downward pointing, but we need to orient $S$ with upward normal vector. For correct orientation, we must choose the normal vector $\vec{n}=v \vec{k}$.

The total magnetic flux through the disk will be:

$$
\begin{aligned}
\iint_{S} \vec{F} \cdot d \vec{S} & =\iint_{S} \vec{F} \cdot \vec{n} d \sigma= \\
& =\iint_{S}\left(\left(x^{2}+y^{2}\right) \vec{k}\right) \cdot(v \vec{k}) \frac{1}{|v|} d \sigma= \\
& =\iint_{D}\left(\left(v^{2} \cos ^{2} u+v^{2} \sin ^{2} u\right) \vec{k}\right) \cdot(v \vec{k}) \frac{1}{|v|} \cdot|v| d P= \\
& =\iint_{D}\left(v^{3}\right) d P= \\
& =\int_{0}^{2 \pi} \int_{0}^{6}\left(v^{3}\right) d v d u=648 \pi
\end{aligned}
$$

Calculating Mass Flow Rate. If $\vec{F}$ represent a velocity field (with units of meters per second) of a fluid with constant density $\rho$, then the mass flow rate of the fluid across the surface $S$ is given with the flux integral $\iint_{S} \rho \vec{F} \cdot \vec{n} d \sigma$. If we want to find the flow rate (measured in volume per time) instead, we can use flux integral $\iint_{S} \vec{F} \cdot \vec{n} d \sigma$, which leaves out the density.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

Calculating Heat Flow. In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point $(x, y, z)$ in an object is $T(x, y, z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $\vec{F}=-k \nabla \vec{T}$, where the constant $k$ is the thermal conductivity of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface $S$ in the object is given by the flux integral

$$
\iint_{S} \vec{F} \cdot d \vec{S}=\iint_{S}-k \nabla \vec{T} d \vec{S}
$$

